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THE NUMERICAL SOLUTION OF SINGULAR SINGULARY-PERTURBED INITIAL --ETC(U)
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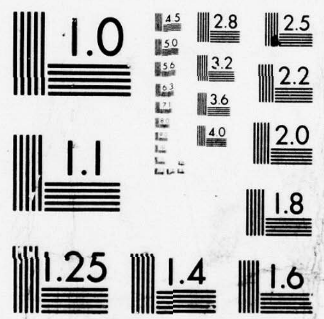

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THE NUMERICAL SOLUTION OF SINGULAR SINGULARLY-PERTURBED INITIAL VALUE PROBLEMS

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ABSTRACT

We consider the vector initial value problem $\varepsilon\dot{y} = f(y,t,\varepsilon)$, $y(0) = y^0(\varepsilon)$ in the situation when the $m \times m$ matrix $f_y(y,t,0)$ is singular with constant rank $k < m$ and has k stable eigenvalues. We show how to determine the unique limiting solution Y_0 of the reduced problem $f(Y_0,t,0) = 0$ and how to obtain a uniform asymptotic expansion of the solution which is valid for small values of ε on finite t intervals. A numerical technique is developed to calculate the limiting solution and the results of some examples are compared with an existing code for stiff differential equations.

1. INTRODUCTION

We consider the initial value problem

$$\varepsilon\dot{y} = f(y,t,\varepsilon) \quad , \; y(0,\varepsilon) = y^0(\varepsilon) \tag{1.1}$$

for m nonlinear differential equations on a finite interval $0 \le t \le T$ in the limit as the small positive parameter ε

tends to zero. Familiarity with singular perturbation problems leads one to expect that (under appropriate stability conditions) the solution of (1.1) would converge to a limiting solution Y_0 of the reduced system

$$f_0(Y_0,t) \equiv f(Y_0,t,0) = 0 \tag{1.2}$$

as $\varepsilon \to 0$, at least away from an initial boundary layer region of nonuniform convergence. For example, in the classical Tikhonov problem (cf. Wasow (1976)), when the Jacobian F_y $f_y(y,t,0)$ has stable eigenvalues for all y and t (the region of stability can be further restricted), then (1.2) has a unique solution $Y_0(t)$ which is the limiting solution of (1.1) for $t > 0$. The solution generally converges nonuniformly at $t = 0$ since there is no reason to expect that $Y_0(0) = y^0(0)$. Indeed, if f is infinitely differentiable in y and t and has an asymptotic expansion in ε then the solution $y(t,\varepsilon)$ of (1.1) can be represented asymptotically in the form

$$y(t,\varepsilon) = Y(t,\varepsilon) + \Pi(\tau,\varepsilon), \tag{1.3}$$

throughout $0 \le t \le T$. The outer solution Y and the boundary layer correction Π both have asymptotic expansions in ε, and Π tends to zero as the stretched (or boundary layer) variable

$$\tau = t/\varepsilon \tag{1.4}$$

tends to infinity.

We wish to consider (1.1) when matrix $f_y(y,t,0)$ is singular, and in particular satisfies:

Hypothesis (H): $f_y(y,t,0)$ *has constant rank k,* $0 \le k < m$ *for all t in* $0 \le t \le T$ *and all y; its nonzero eigenvalues have negative real parts there; and its null space is spanned by m − k linearly independent eigenvectors.*

In this case we will find that the asymptotic solution of

(1.1) still has the form (1.3) whenever the reduced system (1.2) is consistent and solvable, i.e., whenever (1.2) has at least one solution. However, because f_{0y} is singular, (1.2) no longer has a unique solution and additional analysis is necessary to determine the unique limiting solution for $t > 0$. We call such problems "singular singularly-perturbed problems". Two simple scalar examples illustrating some of the possibilities are (i) $\varepsilon\dot{y} = 1$, $y(0) = 0$ and (ii) $\varepsilon\dot{y} = 0$, $y(0) = 0$. For (i), the reduced problem $1 = 0$ is inconsistent, and while $y = \tau = t/\varepsilon$ is a solution of the form (1.3) we see that Π becomes unbounded as $\tau \to \infty$. For (ii), the reduced problem $0 = 0$ is satisfied for all Y_0, but only the trivial solution $Y_0 = 0$ is a limit of the unique solution $y = 0$.

Campbell and Rose (1978) studied constant coefficient singular singularly-perturbed problems of the form

$$\varepsilon\dot{y} = F(\varepsilon)y \tag{1.5}$$

and showed that the "semistability" condition of Hypothesis (H) is a necessary and sufficient condition for a limiting solution to exist for all $t > 0$ and all initial vectors y^0. O'Malley (1978) obtained asymptotic solutions of (1.1) in the almost-linear case when $f(y,t,0) = F(t)y + g(t)$, assuming that the linear reduced system $F(t)Y_0 + g(t) = 0$ is consistent. A preliminary study of nonlinear systems was reported in O'Malley and Flaherty (1976). Additional work on singular singular-perturbed problems was done by Vasil'eva and others (cf. Vasil'eva (1976) and the references contained therein).

Asymptotic solutions with a different structure than (1.3) might result if initial values were restricted. For example, consider (1.5) with

$$F(\varepsilon) = \begin{bmatrix} 0 & -\varepsilon \\ -1 & 0 \end{bmatrix}$$

and suppose that the initial components satisfy $y_1^0 = \sqrt{\varepsilon} y_2^0$, then we have the trivial solution for $t > 0$, but the boundary layer behaviour is determined by the stretched variable $\sigma = t/\sqrt{\varepsilon}$. More complicated limiting solutions would occur if we allowed "turning points" where the rank of $f_y(y,t,0)$ changes at particular y and t values. Studies of these interesting and difficult problems are contained in the work of Howes (1978) and Kreiss (1978). The latter also contains numerical methods. Two simple scalar examples of such problems are $\varepsilon\dot{y} = -y^3 + \varepsilon y$ and $\varepsilon\dot{y} = (t - 1/2)y$, where the ranks of $f_y(y_0,t,0)$ change at $y = 0$ and $t = 1/2$, respectively.

In Section 3 of this paper we develop asymptotic expansions for the outer solutions $Y(t,\varepsilon)$ of a special class of singular singularly-perturbed problems and in Section 4 we consider more general problems. Some preliminary linear algebra is presented in Section 2. Expansions for the boundary layer correction $\Pi(\tau,\varepsilon)$ and a proof of the asymptotic validity of our solutions have also been obtained and will be reported in O'Malley and Flaherty (1978). In Section 5 we develop a numerical procedure for calculating the limiting solution $Y_0(t)$ which is based on the expansion of Section 4 and in Section 6 we apply this procedure to some examples and discuss the results.

Our primary motivation for this work is the need to develop numerical procedures for singularly-perturbed (or stiff) two-point boundary value problems. However, our results should be applicable to initial value problems in power generation and distribution systems, biological and chemical reactions, and electrical networks. A new application is ill-conditioned nonlinear optimization problems (cf. Boggs and Tolle (1977)).

2. ALGEBRAIC PRELIMINARIES

We shall attempt to find an asymptotic solution of (1.1) in the form given by (1.3). The decay of Π as $\tau \to \infty$ implies that the outer solution $Y(t,\varepsilon)$ should satisfy

$$\varepsilon\dot{y} = f(y,t,\varepsilon) \tag{2.1}$$

as a power series in ε, i.e.,

$$Y(t,\varepsilon) \sim \sum_{j=0}^{\infty} Y_j(t)\varepsilon^j. \tag{2.2}$$

Under Hypothesis (H) we are guaranteed that

$$f_{0y}(y,t) \equiv f_y(y,t,0) \equiv \frac{\partial f}{\partial y}(y,t,0) \tag{2.3}$$

can be put into its reduced echelon form by an orthogonal matrix $E(y,t)$. Golub (1965) discussed a numerical procedure for obtaining E by performing a sequence of k Householder transformations. The differentiability of E follows that of f_0 under the constancy of rank condition (cf. Golub and Pereyra (1976)). We partition E as

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{2.4}$$

where E_1 is $k \times m$, E_2 is $(m-k)\times m$, and

$$E_2 f_{0y} = 0 \tag{2.5}$$

In addition,

$$Ef_{0y} E^T = \begin{bmatrix} S & X \\ 0 & 0 \end{bmatrix} \tag{2.6}$$

where

$$S = E_1 f_{0y} E_1^T \quad , \quad X = E_1 f_{0y} E_2^T \quad , \tag{2.7}$$

Hypothesis (H) guarantees that S has k stable eigenvalues. We note that Clasen *et al* (1978) used such constant orthogonal matrices E to integrate stiff problems, while O'Malley (1978) used time dependent matrices for almost linear problems.

The orthogonality of E further implies that

$$E_1 E_2^T = 0 \ , \ E_1 E_1^T = I_k \ , \ E_2 E_2^T = I_{m-k} \ , \text{ and} \tag{2.8}$$

$$E_1^T E_1 + E_2^T E_2 = I_m \ ,$$

where I_m is the $m \times m$ identity matrix. Using the last relation we introduce the complementary orthogonal projections

$$P = E_1^T E_1 \quad , \quad Q = E_2^T E_2 \tag{2.9}$$

which provide a direct sum decomposition of m-space with Q projecting onto $N(f_{0y}^T)$, the null space of f_{0y}^T, and P projecting onto $R(f_{0y})$, the range of f_{0y}.

3. A SPECIAL PROBLEM: E(y,t) = E(t)

In this section we examine special problems (1.1) when the orthogonal matrix E(y,t) introduced in Section 2 is independent of y. This, of course, includes the nearly linear problems where

$$f(y,t,0) = F(t)y + g(t)$$

and "classical" singular perturbation problems having the form

$$\varepsilon \dot{y}_1 = f_1(y_1,y_2,t) + \varepsilon g_1(y_1,y_2,t,\varepsilon)$$

$$\varepsilon \dot{y}_2 = \varepsilon g_2(y_1,y_2,t,\varepsilon) \quad .$$

Here, y_1 is a k-vector, y_2 is an (m-k)-vector, and $\partial f_1/\partial y_1$ is of rank k. In this case $E = I_m$.

We define

$$z = E(t)y, \tag{3.1}$$

and further partition z like E, i.e.,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} E_1 y \\ E_2 y \end{bmatrix}. \tag{3.2}$$

Introducing (3.2) into (1.1) gives the following system for z:

$$\varepsilon \dot{z}_1 = h_1(z_1,z_2,t,\varepsilon) \quad , \quad z_1(0) = E_1(0)y^0 \tag{3.3a}$$

$$\dot{z}_2 = h_2(z_1,z_2,t,\varepsilon)/\varepsilon \quad , \quad z_2(0) = E_2(0)y^0 \tag{3.3b}$$

where,

$$h_i = E_i f(E^T z,t,\varepsilon) + \varepsilon E_i E^T z \quad , \quad i = 1,2. \tag{3.4}$$

We have divided (3.3b) through by ε since

$$h_2(z_1,z_2,t,0) = 0 \tag{3.5}$$

necessarily follows if the reduced system (1.2) for (1.1) is consistent. This is because

$$\frac{\partial h_2}{\partial z_i}(z_1,z_2,t,0) = E_2 f_{0y}(E^T z,t)E_i^T = 0 \quad , \quad i = 1,2$$

upon use of (3.4) and (2.5). Thus, $h_2(z_1,z_2,t,0)$ is a function of t only. However, the reduced system (1.2) implies the corresponding reduced system

$$h_i(z_1,z_2,t,0) = 0 \quad , \quad i = 1,2$$

for (3.3). Hence $h_2(z_1,z_2,t,0)$ must be the trivial function of t on $0 \le t \le T$ for any z_1 and z_2, otherwise (1.2) would have no solutions. Tikhonov's results apply to (3.3) because his stability condition that

$$\frac{\partial h_1}{\partial z_1}(z_1,z_2,t,0) = E_1 f_{0y}(E^T z,t)E_1^T \equiv S \tag{3.6}$$

have stable eigenvalues holds for all z_1, z_2 and for all t on $0 \le t \le T$. Thus, (3.3) has an asymptotic solution of the form

$$\begin{aligned} z_1(t,\varepsilon) &= Z_1(t,\varepsilon) + \Lambda_1(\tau,\varepsilon) \\ z_2(t,\varepsilon) &= Z_2(t,\varepsilon) + \varepsilon\Lambda_2(\tau,\varepsilon) \end{aligned} \tag{3.7}$$

(cf. O'Malley (1974)), where the outer solution (Z_1,Z_2) and the boundary layer correction (Λ_1,Λ_2) each have power series expansions in ε with the boundary layer correction decaying to zero as $\tau = t/\varepsilon \to \infty$.

Since the outer solution provides the asymptotic solution for $t > 0$, we must have

$$\varepsilon\dot{Z}_1 = h_1(Z_1,Z_2,t,\varepsilon) \quad , \quad \dot{Z}_2 = h_2(Z_1,Z_2,t,\varepsilon)/\varepsilon \tag{3.8}$$

satisfied as power series

$$Z_i(t,\varepsilon) \sim \sum_{j=0}^{\infty} Z_{ij}(t)\varepsilon^j \quad , \quad i = 1,2, \tag{3.9}$$

in ε. The leading term must necessarily satisfy the limiting problem

$$h_1(Z_{10},Z_{20},t,0) = 0 \tag{3.10a}$$

$$\dot{Z}_{20} = h_{2_\varepsilon}(Z_{10},Z_{20},t,0) \quad , \quad Z_{20}(0) = E_2(0)y^0(0). \tag{3.10b}$$

Its unique solution is obtained since (3.6) and the implicit function theorem imply that the algebraic equation $h_1(Z_{10},Z_{20},t,0) = 0$ can be uniquely solved for the k-vector

$$Z_{10}(t) = \phi(Z_{20}(t), t) \tag{3.11}$$

leaving the nonlinear (m-k) th order initial value problem

$$\dot{Z}_{20} = \frac{\partial h_2}{\partial \varepsilon}(\phi(Z_{20},t), Z_{20},t,0) \ , \ Z_{20}(0) = E_2(0)y^0(0) \tag{3.12}$$

for Z_{20}. We shall assume that the unique solution to (3.12) continues to exist throughout $0 \le t \le T$. Note that the reduced system (1.2) implied that both $h_1 = 0$ and $h_2 = 0$ along $(Z_{10},Z_{20},t,0)$, but it did not provide equation (3.12) needed to uniquely determine the limiting outer solution (Z_{10},Z_{20}).

Higher order terms in (3.9) satisfy linear problems

$$\frac{\partial h_1}{\partial z_1}(Z_{10},Z_{20},t,0)Z_{1j} + \frac{\partial h_2}{\partial z_2}(Z_{10},Z_{20},t,0)Z_{2j} = g_{1,j-1}(t)$$

$$\dot{Z}_{2j} = \frac{\partial^2 h_2}{\partial z_1 \partial \varepsilon}(Z_{10},Z_{20},t,0)Z_{1j} + \frac{\partial^2 h_2}{\partial z_2 \partial \varepsilon}(Z_{10},Z_{20},t,0)Z_{2j} + \tag{3.13}$$

$$g_{2,j-1}(t), \quad Z_{2j}(0) = -\Lambda_{2,j-1}(0)$$

with the $g_{i,j-1}(t)$'s determined by lower order terms in the outer expansion. One solves the first equation for Z_{1j} as a linear function of Z_{2j}, and then the resulting linear differential equation for Z_{2j}. Thus, the outer expansion (3.9) can be uniquely generated termwise in $0 \le t \le T$ up to a knowledge of the initial value of the boundary layer correction component $\Lambda_2(0,\varepsilon)$.

The boundary layer correction is obtained by noting that both (z_1,z_2) and (Z_1,Z_2) satisfy the differential equations (3.3). Hence, using (3.8) in (3.3) we have

$$\begin{aligned}\frac{d\Lambda_1}{d\tau} &= h_1(Z_1(\varepsilon\tau,\varepsilon) + \Lambda_1(\tau,\varepsilon), Z_2(\varepsilon\tau,\varepsilon) + \varepsilon\Lambda_2(\tau,\varepsilon),\varepsilon\tau,\varepsilon) \\ &\quad -h_1(Z_1(\varepsilon\tau,\varepsilon),Z_2(\varepsilon\tau,\varepsilon),\varepsilon\tau,\varepsilon) \\ \frac{d\Lambda_2}{d\tau} &= [h_2(Z_1(\varepsilon\tau,\varepsilon) + \Lambda_1(\tau,\varepsilon), Z_2(\varepsilon\tau,\varepsilon) + \varepsilon\Lambda_2(\tau,\varepsilon),\varepsilon\tau,\varepsilon) \\ &\quad -h_2(Z_1(\varepsilon\tau,\varepsilon),Z_2(\varepsilon\tau,\varepsilon),\varepsilon\tau,\varepsilon)].\end{aligned} \tag{3.14}$$

We require Λ_1 and Λ_2 to decay as $\tau \to \infty$ and satisfy the initial condition

$$\Lambda_1(0,\varepsilon) = E_1(0)y^0(0) - Z_1(0,\varepsilon). \tag{3.15}$$

Taking

$$\Lambda_i(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \Lambda_{ij}(\tau)\varepsilon^j \qquad i = 1,2 \tag{3.16}$$

we find that the leading term Λ_{10} must satisfy the nonlinear initial value problem

$$\frac{d\Lambda_{10}}{d\tau} = h_1(Z_{10}(0) + \Lambda_{10}(\tau), Z_{20}(0),0,0) \; . \qquad h_1(Z_{10}(0), Z_{10}(0),0,0) \; , \tag{3.17}$$
$$\Lambda_{10}(0) = E_1(0)y^0(0) - Z_{10}(0) \; .$$

This problem has a unique exponentially decaying solution $\Lambda_{10}(\tau)$ since (3.6) implies that $\frac{\partial h_1}{\partial z_1}(z_1, z_2, t, 0)$ has stable eigenvalues for all arguments. Knowing Λ_{10} we can calculate Λ_{20} and successive terms in (3.14). The details of this calculation are omitted here as they will be reported elsewhere (cf. O'Malley and Flaherty (1978)).

The asymptotic validity of the expansion (3.9) follows from Tikhonov's theorem (cf. Wasow (1976) or Vasil'eva and Butuzov (1973)). Returning to the original variables, we have found a unique asymptotic solution of the form (1.3) with the outer solution given by

$$Y(t,\varepsilon) = E^T(t)Z_1(t,\varepsilon) + E_2^T(t)Z_2(t,\varepsilon)$$

and with the exponentially decaying boundary layer correction given by

$$\Pi(\tau,\varepsilon) = E_1^T(\varepsilon\tau)\Lambda_1(\tau,\varepsilon) + \varepsilon E_2^T(\varepsilon\tau)\Lambda_2(\tau,\varepsilon) .$$

The result will even be valid for all $t \geq 0$ provided that Z_{20} decays exponentially as $t \to \infty$ (cf. Hoppensteadt (1966)).

4. THE ORIGINAL PROBLEM

We now return to the original problem where the orthogonal matrix E can depend on y as well as t. As noted in Section 2, the outer solution (2.2) should satisfy the system (2.1) as a power series in ε for $t \geq 0$. The leading term in the expansion will satisfy (1.2) and, for $j \geq 1$, $f_y(Y_0,t,0)Y_j$ will be successively determined by the preceding Y_ℓ, $\ell = 0,1,\ldots, j-1$. This fails to uniquely determine the Y_j's since $f_y(Y_0,t,0)$ has rank $k < m$. We shall instead find them as solutions of differential equations. To this end, we differentiate (2.1) with respect to t to obtain.

$$f_y(Y,t,\varepsilon)\dot{Y} + f_t(Y,t,\varepsilon) = \varepsilon\ddot{Y} \tag{4.1}$$

and use (1.2) and (4.1) together.

We define $E_{10}(t) = E_1(Y_0(t),t)$ and let $E_{20}(t), P_0(t)$, and $Q_0(t)$ be analogously defined. From (2.8) and (2.9) we see that $P_0 + Q_0 = I_m$; thus, we can write

$$\dot{Y} = P_0\dot{Y} + Q_0\dot{Y} \tag{4.2}$$

and seek to obtain equations for $P_0\dot{Y}$ and $Q_0\dot{Y}$. In particular, (4.1) and (4.2) imply

$$E_{10}f_y(Y,t,\varepsilon)(P_0\dot{Y} + Q_0\dot{Y}) = E_{10}(-f_t + \varepsilon\ddot{Y}).$$

Using the stable matrix

$$S_0 = E_{10}f_y(Y_0,t,0)E_{10}^T \tag{4.3}$$

and (2.9) we have

$$\begin{aligned} P_0\dot{Y} = -A_0\{&[f_y(Y,t,\varepsilon) - f_y(Y_0,t,o)]P_0\dot{Y} \\ &+ f_y(Y,t,\varepsilon)Q_0\dot{Y} + f_t(Y,t,\varepsilon) - \varepsilon\ddot{Y}\} \end{aligned} \tag{4.4}$$

where,

$$A_0 = E_{10}^T S_0^{-1} E_{10}. \tag{4.5}$$

From (2.1) we have

$$Q_0\dot{Y} = Q_0 f(Y,t,\varepsilon)/\varepsilon. \tag{4.6}$$

Using (4.3) and (4.6) in (4.2) we find

$$\dot{Y} = -A_0\{[f_y(Y,t,\varepsilon) - f_y(Y_0,t,0)]\dot{Y} + f_t(Y,t,\varepsilon) - \varepsilon\ddot{Y}\} + B_0Q_0f(Y,t,\varepsilon)/\varepsilon$$

where

$$B_0 = I_m - A_0f_{0y}. \tag{4.8}$$

It may be useful to note that B_0 is a projection with

$$B_0P_0 = 0,\ B_0Q_0 = B_0,\ \text{and}\ B_0A_0 = 0.$$

Setting $\varepsilon = 0$ in (4.7) yields the limiting nonlinear equation

$$\dot{Y}_0 = -A_0f_t(Y_0,t,0) + B_0Q_0f_\varepsilon(Y_0,t,0) \tag{4.9}$$

We note that the term $Q_0f_{0y}(Y_0,t)Y_1$ is missing since $Q_0f_{0y} = 0$ upon use of (2.5) and (2.9).

In order to obtain further coefficients Y_j it is necessary to consider the coefficients of higher powers of ε in (4.7). Thus, setting

$$f(Y,t,\varepsilon) \sim \sum_{j=0}^{\infty} f_j(Y,t)\varepsilon^j \tag{4.10}$$

and using the expansion (2.2) for Y implies that

$$\begin{aligned} f(Y,t,\varepsilon) = {} & f_0(Y_0,t) + \varepsilon[f_1(Y_0,t) + f_{0y}(Y_0,t)Y_1] \\ & + \varepsilon^2[f_2(Y_0,t) + f_{0y}(Y_0,t)Y_2 + f_{1y}(Y_0,t)Y_1 \\ & + \frac{1}{2}(f_{0yy}(Y_0,t)Y_1)Y_1] + O(\varepsilon^3) \end{aligned}$$

together with corresponding expansions for $f_t(Y,t,\varepsilon)$ and $f_y(Y,t,\varepsilon)$. The coefficient of ε in (4.7) then provides the following nonlinear equation for Y_1

$$\begin{aligned} \dot{Y}_1 = {} & -A_0\{f_{1t}(Y_0,t) + f_{0ty}(Y_0,t)Y_1 + [f_{1y}(Y_0,t) \\ & + f_{0yy}(Y_0,t)Y_1][P_0\dot{Y}_0 + Q_0f_1(Y_0,t)] - \ddot{Y}_0\} \\ & + B_0Q_0\{f_2(Y_0,t) + f_{1y}(Y_0,t)Y_1 + \frac{1}{2}[f_{0yy}(Y_0,t)Y_1]Y_1\} \end{aligned}$$

Except for the final quadratic term, this is a nonhomogeneous linearization of the equation for Y_0. Higher order terms Y_j, $j \geq 2$, satisfy linear differential equations with successively known nonhomogeneous terms.

We note that it may be advantageous to obtain differential equations for the successive terms Y_j of the outer expansion even in the special case (see section 3) where $E(y,t)$ is independent of y. In that case, we solved the nonlinear algebraic equation (3.10a) for Z_{10} as a function of Z_{20}, followed by a nonlinear initial value problem (3.10b) for Z_{20}. It may often be numerically simpler to solve the initial value problem (4.9) for $Y_0(t)$ and those for later terms successively. We have not, however, fully explored both possibilities.

We will have to assume, of course, that the nonlinear initial value problems, (4.9) and (4.11), for Y_0 and Y_1 have solutions on $0 \leq t \leq T$. Moreover, since (1.2) and its time derivative (4.1) are built into (4.7), consistency with (1.2) at $t = 0$ implies consistency of the outer expansion for $t > 0$. If consistency failed, the form (1.3) of the solution would be inappropriate. Thus, using (1.2), (2.1), (2.2), and (4.10), we must have

$$
\begin{aligned}
&f_0(Y_0(0),0) = 0, \\
&f_{0y}(Y_0(0),0)Y_1(0) = \dot{Y}_0(0) - f_1(Y_0(0),0) \;, \qquad (4.12) \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}
$$

These equations always have a solution under our assumptions. For example, in the second equation we must have $\dot{Y}_0(0) - f_1(Y_0(0),0)$ in the range of $f_{0y}(Y_0(0),0)$. Recall, however, that $I_m = P_0 + Q_0$ provides a direct sum decomposition of m space with

$$R(Q_0) = N(f_{0y}^T(Y_0(0),0)) \quad \text{and} \quad R(P_0) = R(f_{0y}(Y_0(0),0)).$$

Thus, the second of (4.12) will be automatically satisfied since $Q_0 f_{0y} = 0$ implies that $Q_0[\dot{Y}_0(0) - f_1(Y_0(0),0)] = 0$.

Because f_{0y} has rank k, k components of $Y_0(0)$ are determined as a function of the remaining m-k components. Indeed, we could attempt to solve (1.2) for $E_{10}(0)Y_0(0)$ in terms of $E_{20}(0)Y_0(0)$ since $S_0(0)$ (cf. (4.3)) is nonsingular. Likewise, for $j > 0$, termwise determination of $f_{0y}(Y_0(0),0)Y_j(0)$ implies that of $E_{10}(0)Y_j(0)$ (by an argument similar to the one preceding (4.3)). Thus, $E_{10}(0)Y_j(0)$, or $P_0(0)Y_j(0)$, is determined termwise while $E_{20}(0)Y_j(0)$, or $Q_0(0)Y_j(0)$, may be specified. The purpose of the boundary layer correction is to compensate for the jump in $P_0(0)(Y_j(0) - y_j^0)$ and to specify the values of $Q_0(0)Y_j(0)$, $j \geq 0$.

Once again, the representation (1.3) and the fact that the differential equation (1.1) is satisfied by both y and Y imply that the boundary layer correction $\Pi(\tau,\varepsilon)$ must satisfy the nonlinear equation

$$\frac{d\Pi}{d\tau} = f(Y(\varepsilon\tau,\varepsilon) + \Pi(\tau,\varepsilon),\varepsilon\tau,\varepsilon) - f(Y(\varepsilon\tau,\varepsilon),\varepsilon\tau,\varepsilon) \tag{4.13}$$

as a power series

$$\Pi(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\varepsilon^j \tag{4.14}$$

in ε and decay to zero as $\tau \to \infty$. Moreover,

$$\Pi(0,\varepsilon) \sim y^0(\varepsilon) - Y(0,\varepsilon). \tag{4.15}$$

The details of the calculation of the boundary layer correction and a proof of the asymptotic validity of the solution are omitted here and will be presented in O'Malley and Flaherty (1978). We summarize our findings, however, in the following theorem.

Theorem: Consider the initial value problem

$$\varepsilon\dot{y} = f(y,t,\varepsilon) \ , \ y(0) = y^0(\varepsilon)$$

for an m-vector y as $\varepsilon \to 0^+$. Assume that:

(i) f is infinitely differentiable in y and t and f and $y^0(\varepsilon)$ have asymptotic series expansions in powers of ε.

(ii) There exists an infinitely differentiable orthogonal matrix $E(y,t)$ for all y and for t in the interval $0 \le t \le T$ such that $E(y,t)f_y(y,t,0)$ is row-reduced and of rank k, $0 \le k < m$. Moreover, partitioning E after its first k rows as in (2.4), we have

$$Ef_y(y,t,0)E^T = \begin{bmatrix} S & X \\ 0 & 0 \end{bmatrix}$$

where $S = E_1 f_y(y,t,0)E_1^T$ is a stable matrix for all values of y and t.

(iii) The nonlinear system

$$f(Y_0(0),0,0) = 0 \tag{4.16a}$$

$$Q(Y_0(0),0)[y^0(0)-Y_0(0) + \int_0^\infty f(Y_0(0) + \Pi_0(\tau),0,0)d\tau] = 0 \tag{4.16b}$$

can be uniquely solved for $Y_0(0)$. Here, $\Pi_0(\tau)$ is the decaying solution of

$$\frac{d\Pi_0}{d\tau} = f(Y_0(0) + \Pi_0(\tau),0,0) - f(Y_0(0),0,0),$$
$$\Pi_0(0) = y^0(0) - Y_0(0).$$

(iv) The matrix

$$I - C_0E_{10}(0)B_0(Y_0(0),0)$$

is invertible for a particular matrix C_0. (This insures that $Y_1(0)$ may be uniquely determined.)

(v) The initial value problems (4.9) and (4.11) have solutions on the interval $0 \le t \le T$.

Then, the initial value problem (1.1) has a unique solution of the form

$$y(t,\varepsilon) = Y(t,\varepsilon) + \Pi(\tau,\varepsilon).$$

Some comments on this theorem are listed below.

(i) Hypothesis (ii) is guaranteed by out earlier Hypothesis (H).

(ii) The theorem is easily obtained from Tikhonov's result if $E(y,t)$ is independent of y. It is considerably simplified if only $E_2(Y_0(0),0)$, and thereby $Q(Y_0(0),0)$, is independent of $Y_0(0)$. In this case (4.16b) reduces to the linear equation

$$Q_0(0)[y^0(0) - Y_0(0)] = 0 \qquad (4.17)$$

and (4.16a) becomes a nonlinear equation for $P_0(0)Y_0(0)$. It can be further shown that the invertibility condition of Hypothesis (iv) then will be automatically satisfied.

(iii) Higher order terms follow without complication under these hypotheses.

(iv) Vasil'eva (1976) considers such problems under a list of ten hypotheses, generally paralleling, but more restrictive than ours. Her most critical assumption involves the existence of a k-dimensional manifold of decaying solutions for $\Pi_0(\tau)$ which can, more or less, be stated in the form

$$E_{20}(0)\Pi_0(\tau) = \Phi(E_{10}(0)\Pi_0(\tau))$$

for a particular function Φ and for all τ and $Y_0(0)$. At $\tau = 0$, we would have

$$E_{20}(0)[y^0(0) - Y_0(0)] = \Phi(E_{10}(0)[y^0(0) - Y_0(0)])$$

where $Y_0(0)$ must also satisfy the reduced equation at $\tau = 0$. This analog of Hypothesis (iii) should uniquely determine $Y_0(0)$ so that the resulting $\Pi_0(0)$ lies on the manifold of initial values corresponding to decaying solutions of $\Pi_0(\tau)$.

5. NUMERICAL ALGORITHM

We have developed an algorithm based on the asymptotic analysis of Section 4 to calculate the leading term $Y_0(t)$ in the outer solution. For most problems it is possible to calculate numerical solutions without explicitly identifying a small parameter ε; thus, we consider initial value problems in the form

$$\dot{y} = \hat{f}(y,t,\varepsilon) \equiv f(y,t,\varepsilon)/\varepsilon \ , \ y(0) = y^{\cdot}(\varepsilon). \tag{5.1}$$

The ε, although shown in (5.1), is to be regarded as unidentifiable. However, if the actual limiting solution is desired, the evaluation of $\dot{y}$ in (5.1) causes overflow, or the order ε terms in f are close to the unit roundoff of the computer relative to the order unity terms in f, then a value of ε can be identified and the initial value problem can be written in the form of equation (1.1). The actual computer code is capable of handling both cases, and all that the user need do is define $\hat{f}$ as in (5.1) or f as in (1.1).

The algorithm consists of two parts: (i) calculating the initial conditions $Y_0(0)$ for the outer problem and (ii) integrating the differential equation (cf., (4.9)) for $Y_0(t)$. We first describe the integration procedure.

The differential equation (4.9) for Y_0 is not stiff; hence, any good code for integrating non-stiff initial value problems may be used. We use both the Adams' methods that are incorporated into the Hindmarsh (1974) version of Gear's code and the IMSL version of the Bulirsch and Stoer (1966) extrapolation procedure. Both of these codes require the evaluation of $\dot{Y}_0$ as a function of Y_0 and t, and we accomplish this as follows:

(i) Calculate $E(Y_0,t)$ by decomposing $\hat{f}_y(Y_0,t,\varepsilon)$. It is not necessary to set ε to 0 unless ε has been explicitly recognized and the actual limiting solution is desired. Golub's (1965) procedure, which uses a sequence of k Householder transformations with column pivoting, is used to obtain E. At the ν *th* step, $1 \leq \nu \leq k$, of this procedure we have

$$E^{(\nu)}(Y_0,t)\hat{f}_y(Y_0,t,\varepsilon) = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} K^T$$

where, U is $\nu \times \nu$ and upper triangular, V is $\nu \times (m-\nu)$, and K is a permutation matrix due to the column pivoting. The procedure terminates, and the rank k of $\hat{f}_y$ is determined, when the maximum available pivot element in W is small relative to the diagonal elements of U. We then have $E = E^{(k)}$. The decomposition is not performed at every time step, but, rather a test is used to determine if E has changed by too much. Thus, the same matrix E may be used for several time steps or, when E is constant, for the entire integration. If at any stage of the computation the rank k of $\hat{f}_y$ changes, a turning point has probably been encountered, and the integration is terminated.

(ii) Partition E into E_1 and E_2 as in (2.4). Calculate

$Q = E_2^T E_2$ and $S = E_1 \hat{f}_y(Y_0,t,\varepsilon)E_1^T$.

(iii) Calculate $Q\dot{Y}_0 = Q\hat{f}(Y_0,t,\varepsilon)$ and $b = -E_1[\hat{f}_t(Y_0,t,\varepsilon) + \hat{f}_y(Y_0,t,\varepsilon)(Q\dot{Y}_0)]$. When ε is explicitly recognized $Q\dot{Y}_0$ is calculated as $Qf_\varepsilon(Y_0,t,0)$.

(iv) Solve $S(E_1\dot{Y}_0) = b$ for $E_1\dot{Y}_0$ by Gaussian elimination and calculate

$$P\dot{Y}_0 = E_1^T(E_1\dot{Y}_0).$$

(v) Calculate $\dot{Y}_0 = P\dot{Y}_0 + Q\dot{Y}_0$.

We now turn to the calculation of the initial conditions $Y_0(0)$ for the outer problem. This is a difficult task when E_2 depends on y. It requires the solution of the nonlinear system (4.16) and the computation of the boundary layer solution $\Pi_0(\tau)$, which itself depends on $Y_0(0)$. It is possible that the integral in (4.16b) may be adequately approximated by a very crude quadrature rule, which would greatly simplify the computation. Miranker (1973) has successfully used such a technique on stiff problems, but we have not as yet explored this possibility. Our code has only been implemented for problems where E_2 is independent of y; thus, when ε is not explicitly recognized $Y_0(0)$ is determined as the solution of

$$\begin{aligned} &\hat{f}(Y_0(0),0,\varepsilon) = 0 \\ &E_2[Y_0(0) - y^0(\varepsilon)] = 0. \end{aligned} \tag{5.2}$$

A Newton like iteration scheme, which closely parallels the computation of $\dot{Y}_0(t)$ is used to solve this nonlinear system. The procedure is outlined below.

(i) Select an initial guess $X^{(0)}$ for $Y_0(0)$, e.g., $X^{(0)} = y^0$ and set $\mu = 0$.

(ii) Calculate $E^{(\mu)}$ by decomposing $\hat{f}_y(X^{(\mu)},0,\varepsilon)$. This is performed as in step (i) of the procedure for calculating $\dot{Y}_0$. If E_1 is independent of y then this step need only be performed once.

(iii) Calculate $Q^{(\mu)}$ and $S^{(\mu)}$ as in step (ii) of the previous procedure.

(iv) Calculate $q^{(\mu+1)} = Q^{(\mu)}(y^0 - X^{(\mu)})$ and

$$b^{(\mu)} = -E_1^{(\mu)}[\hat{f}(X^{(\mu)},0,\varepsilon) + \hat{f}_y(X^{(\mu)},0,\varepsilon)q^{(\mu+1)}]$$

(v) Solve $S^{(\mu)}[E_1^{(\mu)}(X^{(\mu+1)} - X^{(\mu)})] = b^{(\mu)}$ for $E_1^{(\mu)}(X^{(\mu+1)} - X^{(\mu)})$ and calculate $p^{(\mu+1)} = (E_1^{(\mu)})^T[E_1^{(\mu)}(X^{(\mu+1)} - X^{(\mu)})]$

(vi) Set $X^{(\mu+1)} = X^{(\mu)} + p^{(\mu+1)} + q^{(\mu+1)}$.

If $\| X^{(\mu+1)} - X^{(\mu)} \|$ is less than some prescribed tolerance set $Y_0(0) = X^{(\mu+1)}$, otherwise increment μ by 1 and repeat steps (ii) through (vi).

Of course, if the problem is almost linear then only one iteration need be performed.

The entire procedure was successfully applied to several examples, some of which are discussed in the next section.

6. NUMERICAL EXAMPLES AND DISCUSSIONS OF RESULTS

In this section we present and discuss the results of three examples comparing our method of Section 5 with Hindmarsh's (1974) version of Gear's code for stiff differential equations. Both the Adams' methods that we use to integrate the reduced differential equation and Gear's stiffly stable methods are contained in this code, and the user sets a parameter to select the appropriate method. Hindmarsh's code and the IMSL Bulirsch and Stoer code also require the user to select an estimate for the relative local discretization error and an initial step size for the integration. In all cases we selected the relative error tolerance as 10^{-6}. This is perhaps a bit too severe for our methods, because if the problem is not very stiff the reduced solution will be calculated with more accuracy than necessary. The initial step size was selected as 10^{-4} for Adams' methods, $1/10\varepsilon$ for Gear's methods, and 1 for Bulirsch and Stoer's method. We found that the IMSL code was extremely sensitive to the choice of the initial step size and the times for the integration varied quite dramatically depending on this choice. Our choice of unity seemed near optimal for the problems that we considered.

In the tables that follow five numerical solutions are compared. The solutions labeled "asymptotic" were calculated by our method without explicitly recognizing ε and using either the Hindmarsh (Adams) or the IMSL codes; those labeled "Gear" were solved by Hindmarsh's (Gear) code; and those labeled "reduced" were calculated by our method with ε explicitly set to zero. Additional headings in the tables are as follows:

e is the maximum difference in any component, times 10^6, between a numerical solution and the exact solution at terminal time T. For our asymptotic or limiting solutions

$$e = \max_{1\le i\le m} |Y_{0i}(T) - y_i(T)| \times 10^6.$$

In general, for the examples considered, the error was fairly constant outside of the initial boundary layer.

d when the exact solution is not known, we have tabulated the maximum difference in any component, times 10^6, between solutions obtained by our method and those by Gear's code at terminal time T.

NFE For our asymptotic or limiting solutions this denotes the number of times that $\dot{Y}_0$ was evaluated during the course of the integration. For Gear's solutions it denotes the number of times that $\dot{y}$ was evaluated.

NJE For Gear's solutions this denotes the number of times that $\hat{f}_y$ was evaluated during the course of the integration. Our methods evaluate $\hat{f}_y$ each time $\dot{Y}_0$ is evaluated.

CP Time in milli-seconds to integrate the problem, excluding input/output and supervisor state time. Except where noted it includes the time necessary to calculate the initial conditions $Y_0(0)$ by our method. In most cases the times are averaged over several runs. All calculations were performed on an IBM 360/67 at the Rensselaer Polytechnic Institute.

CP_{rel} CP time relative to the fastest execution time.

The individual examples are discussed below.

Example 1:

$$\dot{y} = \begin{bmatrix} (1/\varepsilon-1) & 2(1/\varepsilon-1) \\ -(1/\varepsilon-1) & -(2/\varepsilon-1) \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad 0 \le t \le T = 1.$$

This constant coefficient example is an adaptation of one considered by Gear (1971). The exact solution is

$$y(t) = \begin{bmatrix} 4e^{-t} & -3e^{-t/\varepsilon} \\ -2e^{-t} & +3e^{-t/\varepsilon} \end{bmatrix}$$

The results are compared in Table 1 for $\varepsilon = 10^{-i}$, i=2,4,6,8. They are typical of the results of subsequent examples in that they show that the accuracy of our method increases as the stiffness increases without an increase in computational effort. On the average, our asymptotic and reduced solutions required 5 milli-seconds to calculate the initial conditions for the outer problem.

TABLE I

Summary of Results for Example 1

METHOD		ε 10^{-2}	10^{-4}	10^{-6}	10^{-8}
Asymptotic (Adams)	e	11100.	111.	1.81	.0130
	NFE	34	34	34	34
	CP	135	137	138	129
	CP_{rel}	1.57	1.60	1.62	1.51
Asymptotic (IMSL)	e	11100.	110.	1.10	.00171
	NFE	33	33	33	33
	CP	89.1	89.8	91.0	89.8
	CP_{rel}	1.04	1.05	1.05	1.05
Reduced (Adams)	e	.126	.701	.701	.701
	NFE	34			
	CP	135			
	CP_{rel}	1.58			
Reduced (IMSL)	e	.000309	.000309	.000309	.000309
	NFE	33			
	CP	85.5			
	CP_{rel}	1.00			
Gear	e	3.51	7.61	8.85	4.40
	NFE	158	183	188	195
	NJE	17	24	25	27
	CP	332	396	407	422
	CP_{rel}	3.89	4.63	4.75	4.94

TABLE II

Summary of Results for Example 2

METHOD		ε 10^{-2}	10^{-4}	10^{-6}	10^{-8}
Asymptotic (Adams)	e	300.	1.98	.640	.666
	NFE	46	46	46	46
	CP	196	192	188	196
	CP_{rel}	1.36	1.33	1.31	1.37
Asymptotic (IMSL)	e	301.	2.65	.0284	.00223
	NFE	49	49	49	49
	CP	154	150	152	148
	CP_{rel}	1.07	1.05	1.06	1.03
Reduced (Adams)	e	214.	2.81	.688	.667
	NFE	46			
	CP	187			
	CP_{rel}	1.30			
Reduced (IMSL)	e	214.	2.14	.0201	.00175
	NFE	49			
	CP	144			
	CP_{rel}	1.00			
Gear	e	.541	1.26	3.93	3.97
	NFE	167	191	196	203
	NJE	20	25	26	28
	CP	341	399	406	421
	CP_{rel}	2.38	2.78	2.83	2.93

TABLE III

Summary of Results for Example 3

METHOD		ε 10^{-2}	10^{-4}	10^{-6}	10^{-8}
Asymptotic (Adams)	d	317.	3.39	.352	.358
	NFE	30	30	30	30
	CP	170	170	170	172
	CP_{rel}	1.66	1.66	1.66	1.68
Asymptotic (IMSL)	d	317.	3.56	.440	.446
	NFE	21	21	21	21
	CP	108	107	107	107
	CP_{rel}	1.05	1.05	1.05	1.05
Reduced (Adams)	d	1217.	12.0	.269	.358
	NFE	30			
	CP	165			
	CP_{rel}	1.61			
Reduced (IMSL)	d	1217	12.1	.411	.445
	NFE	21			
	CP	102			
	CP_{rel}	1.00			
Gear	NFE	143	144	150	158
	NJE	17	19	21	23
	CP	343	365	380	400
	CP_{rel}	3.35	3.57	3.72	3.91

TABLE IV

Time to integrate from t=0 to t = T = 10ε for Example 3

ε	Asymptotic (Adams) NFE	CP	Gear NFE	NJE	CP	d	CP_{ratio}
10^{-2}	12	82.9	119	14	316	451.	3.81
10^{-4}	5	52.1	121	15	341	4.81	6.54
10^{-6}	2	37.1	121	15	342	.0361	9.22
10^{-8}	2	37.4	121	15	341	.0231	9.12

TABLE V

Time to integrate from t=0 to t = T = 1 for Example 3 using initial conditions for the outer problem (results for ε = 0 are the reduced solution)

	Asymptotic/ Reduced (Adams)			Asymptotic/ Reduced (IMSL)			Gear			
ε	NFE	CP	CP_{rel}	NFE	CP	CP_{rel}	NFE	NJE	CP	CP_{rel}
10^{-2}	30	142	1.87	21	79	1.04	54	6	103	1.36
10^{-4}	30	141	1.87	21	79	1.04	56	5	102	1.35
10^{-6}	30	141	1.87	21	79	1.04	43	8	100	1.32
10^{-8}	30	143	1.89	21	78	1.04	51	10	118	1.55
0	30	138	1.83	21	76	1.00				

Example 2:

$$\dot{y} = \frac{1}{\varepsilon}\begin{bmatrix} (y_1+y_2)[1 - \frac{1}{2}(y_1+y_2)^2] - \frac{\varepsilon}{\sqrt{2}}(y_2^2-y_1^2) \\ (y_1+y_2)[1 - \frac{1}{2}(y_1+y_2)^2] + \frac{\varepsilon}{\sqrt{2}}(y_2^2-y_1^2) \end{bmatrix}, \quad y(0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$0 \le t \le T = 2$$

This nonlinear problem was contrived so that the orthogonal matrix E is constant and the exact solution is known as

$$y_1(t) = (\xi - \eta)/\sqrt{2} \qquad y_2(t) = (\xi + \eta)/\sqrt{2}$$

with

$$\xi = -(1 - \tfrac{1}{2}e^{-2t/\varepsilon})^{-1/2} \qquad \eta = \sqrt{2}\, e^{-t}\left(\frac{1-1/\xi}{1+1/\sqrt{2}}\right)^{-\varepsilon}$$

The results are presented in Table 2 and generally parallel those for Example 1. The average time required to calculate the initial conditions for the asymptotic and reduced solutions was 24 milli-seconds

Example 3:

$$\dot{y} = \hat{f}(y,\varepsilon) = \frac{1}{\varepsilon}\begin{bmatrix} (y_2^2 - y_1 y_3) - \varepsilon y \\ 2(y_1 y_3 - y_2^2) + \varepsilon y \\ (y_2^2 - y_1 y_3) \end{bmatrix}, y(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ 0 \le t \le T = 1$$

This example arises in chemical reactions and was studied by Vasil'eva (1976). She did not specify the εy_1 terms in $\hat{f}$ nor the initial conditions and they were selected by us rather arbitrarily. The Jacobian $\hat{f}_y(y,0)$ of this system has rank 1 for all $y \neq 0$ and it may be row-reduced by a constant orthogonal matrix E. The results of this example are presented in Table 3. The average time required to calculate the initial conditions for the asymptotic and reduced solutions was 28 milli-seconds.

Our method is to be used on problems where the boundary layer solution is not of interest; hence, we should be able to calculate the initial conditions for the outer problem faster than a stiff differential equation solver could integrate through the boundary layer. In order to provide some evidence that this is the case we solved Example 3 in the interval $0 \le t \le 10\varepsilon$ (the approximate boundary layer region) using Gear's methods and our asymptotic method with the Adams' integrators. The results are presented in Table 4 for $\varepsilon = 10^{-i}$, $i = 2,4,6,8$. The CP times for our method includes both the times to calculate the initial conditions and to integrate the outer problem from $t = 0$ to 10ε. To make the comparison somewhat more fair we re-evaluated E after each iteration, even though it is constant for this example. For $\varepsilon \leq 10^{-6}$ we see that our method can calculate the solution at the edge of the boundary layer region approximately 9 times faster than Gear's methods.

A comparison of the results in Tables 3 and 4 shows that about 90% of the time required to integrate Example 3 from $t = 0$ to 1 by Gear's code is devoted to the boundary layer region for $\varepsilon \leq 10^{-4}$. This suggests the possibility of using our method to calculate the initial conditions for the outer problem and then using a stiff method to integrate the original differential equation. This test was performed on Example 3, and the results are reported in Table 5. All methods use the same initial conditions, i.e., those generated by our method. The CP times required to calculate these conditions are not included in Table 5. The difference between any two computed solutions is less than 3×10^{-4}. While the results are far from conclusive, they do show the extra computational effort that is required by Gear's method for very stiff problems.

The state of the art of numerical methods for stiff initial value problems for ordinary differential equations is very well developed (cf. Enright *et al* (1975)) and a variety of good techniques exist. Nevertheless, there are many problems, particularly in chemical reactions, where asymptotic methods should be useful. They may be used to calculate accurate solutions of very stiff problems, to furnish initial conditions for standard stiff integration routines, and/or as an analytical tool to provide qualitative information about the solutions of stiff problems. In future papers we hope to extend our calculations to initial value problems where E depends on y and to consider boundary value problems.

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We consider the vector initial value problem $\varepsilon\dot{y} = f(y,t,\varepsilon)$, $y(0) = y^0(\varepsilon)$ in the situation when the $m \times m$ matrix $f_y(y,t,0)$ is singular with constant rank $k < m$ and has k stable eigenvalues. We show how to determine the unique limiting solution Y_0 of the reduced problem $f(Y_0,t,0) = 0$ and how to obtain (over please)

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20. Abstract continued.

epsilon

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a uniform asymptotic expansion of the solution which is valid for small values of ε on finite t intervals. A numerical technique is developed to calculate the limiting solution and the results of some examples are compared with an existing code for stiff differential equations.